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A NEW CLASS OF EXACT SOLUTIONS WITH SHOCK WAVES IN GAS DYNAMICS *

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New exact solutions of the equation of one-dimensional gas dynamics with strong shock waves propagating in a moving medium are obtained. The gas flow behind a discontinuity is described by a solution with uniform deformation (see /1, 2/). Solutions of the explosion problem without a counterpressure in a uniformly expanding (or compressing) gas with an arbitrary adiabatic exponent and a non-uniform initial density distribution are constructed, as well as of the problem of cavity collapse in a dust cloud with the formation of a shock wave.

The solution (see /1, 2/) was joined with the shock and detonation waves propagating in a quiescent gas in /3 - 6/. The problem of joining, by the use of the shock wave, of a solution for a moving selfgravitating medium with zero pressure, and the problem of selfsimilar solutions were discussed in /7/. An exact solution of the problem of a strong explosion in a uniformly expanding (or compressing) gas with a special adiabatic exponent equal to 5/3 was obtained in /8/.

1. The exact particular solution of a system of equations, which describes the one-dimensional adiabatic motion of an ideal gas, found by L.I. Sedov, /1, 2/, can be represented by the formulae

$$r = R(t)\,\xi,\ dR = \pm\,[2\varepsilon\lambda^{-1}(R^{-\lambda} + A)]^{1/2}\,dt,\ \lambda = \nu(\gamma - 1) \tag{1.1}$$

$$v = R^{-1}R^{\cdot}r,\ \ p = \frac{p_0(\xi)}{R^{\nu\gamma}},\ \ \rho = \frac{\rho_0(\xi)}{R^{\nu}} \tag{1.2}$$

$$p_0(\xi) = p_0(\xi_0)\exp\int_{\xi_0}^{\xi}\frac{\varepsilon\xi\,d\xi}{G(\xi)},\ \ \rho_0(\xi) = \frac{p_0(\xi)}{G(\xi)}$$

(the dot denotes a derivative with respect to time t).

Here r and ξ are the Euler and Lagrange coordinates, v is the velocity, p is the pressure, ρ is the density, γ denotes the adiabatic exponent $(\gamma > 1)$, $\nu = 1, 2, 3$ for motions with plane, cylindrical and spherical waves respectively, A, ε, ξ_0, $p_0(\xi_0)$ are arbitrary constants, and $\xi_0 > 0$, $p_0(\xi_0) > 0$, $G(\xi)$ is an arbitrary function. By correctly selecting the Lagrangian coordinates we can have $\varepsilon = \pm 1$.

Let us consider the problem of joining the solution of (1.1), (1.2) with a shock wave which propagates in a gas with zero pressure (in a dust medium). We write the conditions at the discontinuity denoting the quantities in front of the shock wave by the index 1, taking into account that $p_1 = 0$ and using relations (1.1) and (1.2):

$$\rho_1 = \frac{\gamma - 1}{\gamma + 1}\frac{p_0(\xi_0)}{R^{\nu}G(\xi_*)}\exp\int_{\xi_0}^{\xi_*}\frac{\varepsilon\xi\,d\xi}{G(\xi)} \tag{1.3}$$

$$\xi_*^{\cdot} = \left[\frac{\gamma - 1}{2}R^{-\lambda-2}G(\xi_*)\right]^{1/2},\ \ G(\xi_*) = \frac{\gamma - 1}{2}R^{\lambda}(R^{\cdot}\xi_* - v_1)^2 \tag{1.4}$$

($\xi_* = r_*/R$ is a Lagrange coordinate of gas particles which occur at the shock wave front, and r_* is the shock wave radius).

The motion of the dust medium before a discontinuity is determined by the relation

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$$r = vt + f(v), \quad \rho = \frac{h(v)}{[vt + f(v)]^{\nu-1}[f'(v) + t]} \tag{1.5}$$

where $f(v)$ and $h(v)$ are arbitrary functions (generally speaking non-unique, and not determined along the whole of the v axis).

Let a shock wave emerge from the centre of symmetry (axis or plane) at the time t'. The following two formulations of the problem of joining the solutions (1.1), (1.2) with (1.5), using a jump, are possible.

Problem A. If the density and pressure distributions behind the shock wave are given, that is the function $G(\xi)$ is known, the integration of the first Eq.(1.4) gives us the law of motion of the shock wave $\xi_*(t)$

$$\int_0^{\xi_*(t)} \frac{d\xi}{[G(\xi)]^{1/2}} = \int_{t'}^{t} \left[\frac{\gamma-1}{2} R^{-\lambda-2}\right]^{1/2} dt \tag{1.6}$$

(the function $R(t)$ is determined from (1.1)). Subsequently, solving the second Eq.(1.4) for v_1, we obtain the dust velocity $v_1(t)$ at the shock wave front at the instant t. After inverting this function from the first Eq.(1.5) we have

$$f(v_1) = r_*[t(v_1)] - v_1 t(v_1)$$

Then the function $h(v)$ is determined from Eq.(1.3) and the second Eq.(1.5).

We note that a similar approach was proposed in /7/.

Problem B. Let the function $f(v)$ be known, that is the velocity distribution in the dust medium is specified. On solving the relation $R\xi_* = v_1 t + f(v_1)$ for v_1, and substituting the expression found into (1.4), we obtain an ordinary first-order differential equation for $\xi_*(t)$. Generally, this equation cannot be integrated by quadratures. However, sometimes integration can be performed here for a specially selected $f(v)$.

2. We select the function $f(v)$ in the form $f(v) = r_0 - vt_0$. Obviously, we can assume without loss of generality that $t_0 = 0,\ r_0 = 0$. Then from the first equation of (1.5) we obtain

$$v = r/t \tag{2.1}$$

Such a velocity distribution for $t < 0$ corresponds to a uniform compression of the medium, and for $t > 0$ to its uniform scattering. At the instant $t = 0$, the whole substance concentrates at the origin of coordinates $(r = 0)$.

At the front of a shock wave we have $v_1 = Rt^{-1}\xi_*,\ v_2 = R^{\cdot}\xi_*$. Since it is necessary that $v_2 > v_1$, the inequality

$$R^{\cdot} > R/t \tag{2.2}$$

should hold after formation of a discontinuity for $t > t'$.

Allowing for (2.1) and (2.2), we transform relation (1.4) to the form

$$G = \frac{\gamma-1}{2} R^{\lambda}\xi_*^2\left(R^{\cdot} - \frac{R}{t}\right)^2, \quad \frac{d}{dt}\ln\xi_* = \frac{\gamma-1}{2R}\left(R^{\cdot} - \frac{R}{t}\right) \tag{2.3}$$

This equation is easy to integrate

$$\xi_* = k\,|R/t|^{(\gamma-1)/2}, \quad k = \text{const} \tag{2.4}$$

The inequality $\xi_*(t') = 0$ holds if $R(t') = 0$, which is possible only when $\varepsilon = +1$. Thus, in the solutions described the gas pressure behind the shock wave always decreases as one approaches the centre of symmetry.

First, let us consider the case when $t' > 0$. We note that it is necessary to have $A > 0$ in (1.1), in order to satisfy condition (2.2).

We introduce the new variables Ω and τ:

$$\Omega = A^{1/\lambda}R, \quad \tau = \sqrt{2\varepsilon/\lambda}\,A^{1/\lambda+1/2}t \tag{2.5}$$

Then, for $\tau > \tau'$, where τ' is a dimensionless instant of time of the appearance of the discontinuity, relations (1.1) and (1.2) can be written as

$$d\Omega = (\Omega^{-\lambda} + 1)^{1/2}d\tau, \quad \tau > \Omega(\Omega^{-\lambda} + 1)^{-1/2} \tag{2.6}$$

Let us estimate as $\Omega \to \infty$ the value of the quantities on both sides of the above inequality assuming that $\lambda \neq 1$. Since $\Omega(\tau') = 0$,

$$\tau = \int_0^{\Omega} (\Omega^{-\lambda} + 1)^{-1/2}\,d\Omega + \tau' = \Omega - \frac{\Omega^{1-\lambda}}{2(1-\lambda)} + O(\Omega^{1-2\lambda})$$

$$\Omega(\Omega^{-\lambda} + 1)^{-1/2} = \Omega - {}^1/_2\,\Omega^{1-\lambda} + O(\Omega^{1-2\lambda})$$

We may now note that if $0<\lambda<1$, then the inequality in (2.6) does not hold for sufficiently large values of Ω. It can be shown that for $\lambda=1$ the inequality is violated. In the solutions discussed, therefore, certainly $\lambda>1$ $(\gamma>(\nu+1)/\nu)$. For values of γ that satisfy this requirement, the graph of the function $\Omega(\tau)$ has the asymptote $\Omega=\tau-\tau_0$ so that

$$\lim_{\tau\to\infty}\left[\frac{d\Omega(\tau)}{d\tau}(\tau-\tau_0)-\Omega(\tau)\right]=0$$

Obviously, to satisfy the inequality in (2.6) for any $\tau>\tau'$ it is necessary that

$$\tau'-\tau_0=\int_0^\infty\frac{\lambda\Omega^{-\lambda}}{2(\Omega^{-\lambda}+1)^{3/2}}\,d\Omega=I$$

It follows that τ' should not be less than the integral in the last equality. Henceforth, we limit ourselves to considering the case of the strict inequality $\tau'>I$.

By (2.4) and (1.1), ξ_* can increase to a certain final value which we denote by $\xi_{\max}$. Thus, we have here the effect of 'freezing' of the shock wave in the dispersing medium. Only those particles whose distance from the centre of symmetry at the instant t' does not exceed $r_{\max}=(2\varepsilon\lambda^{-1}A)^{1/2}\xi_{\max}t'$ pass through a discontinuity. The sphere which contains these particles may be a surface of a contact discontinuity (for example, a boundary with a vacuum).

Henceforth, we shall denote the value of Ω which corresponds to the Lagrangian coordinate of the shock wave ξ_* by Ω_*. Then relation (2.4) can be written in the form

$$\Xi_*=[\Omega_*/\tau(\Omega_*)]^{(\gamma-1)/2}\ (\Xi_*=\xi_*/\xi_{\max}) \tag{2.7}$$

The pressure and density distributions in the gas behind the shock wave are determined from the expressions

$$P_0(\Xi_*)=\frac{p_0(\xi_*)}{p_0(\xi_{\max})}=\exp\int_\infty^{\Omega_*}\frac{\lambda\,d\Omega}{2\Omega^{\lambda+1}\sqrt{\Omega^{-\lambda}+1}\left[\sqrt{\Omega^{-\lambda}+1}-\Omega/\tau(\Omega)\right]} \tag{2.8}$$

$$p(\xi_*,\tau)=A^{\gamma/(\gamma-1)}p_0(\xi_*)/\Omega^{\nu\gamma}(\tau),\quad \tau>\tau(\Omega_*),\quad p_0(\xi_{\max})=\text{const}$$

$$\Lambda_0(\Xi_*)=\frac{\varepsilon\xi_{\max}^2\rho_0(\xi_*)}{p_0(\xi_{\max})}=\nu\frac{P_0(\Xi_*)}{\Xi_*^2}\frac{\Omega_*^{-\lambda}}{\left[\sqrt{\Omega_*^{-\lambda}+1}-\Omega_*/\tau(\Omega_*)\right]^2} \tag{2.9}$$

$$\rho(\xi_*,\tau)=A^{1/(\gamma-1)}\rho_0(\xi_*)/\Omega^\nu(\tau),\quad \tau>\tau(\Omega_*)$$

We will show that the shock wave originates at the centre of symmetry as a consequence of the emission of some positive energy E_0 at the instant τ'. To do this we consider the asymptotic behaviour of the solution as $\tau\to\tau'$ $(\Xi_*\to 0)$. From formulae (2.7)-(2.9) we have (to within small quantities of higher order)

$$\Xi_*=c_1\Omega_*^{(\gamma-1)/2},\quad P_0(\Xi_*)=c_2\Xi_*^{\nu},\quad \Lambda_0(\Xi_*)=\nu c_2\Xi_*^{\nu-2} \tag{2.10}$$

where the constants c_1 and c_2 are determined uniquely by ν, γ and τ'.

The total energy of the gas behind the shock wave is

$$E_2=\int_0^{r_*}\sigma_\nu r^{\nu-1}\left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}\right)dr=c\int_0^{\Xi_*}\Xi_{**}^{\nu-1}\Lambda_0(\Xi_{**})\left[\frac{(\Omega_*^{-\lambda}-1)\,\Xi_{**}}{\nu}+\frac{\Omega_*^{-\lambda}P_0(\Xi_{**})}{\Lambda_0(\Xi_{**})}\right]d\Xi_{**}$$

$$\sigma_\nu=\frac{1}{2}[4\pi(\nu-1)+(\nu-2)(\nu-3)],\quad c=\frac{\sigma_\nu A p_0(\xi_{\max})\,\xi_{\max}^\nu}{\gamma-1}$$

The same mass of gas had the following energy before passing through the shock wave:

$$E_1=\int_0^{\xi_*}\sigma_\nu\xi^{\nu-1}\frac{\rho_0 v_1^2}{2}\,d\xi=\frac{c}{\nu}\int_0^{\Xi_*}\Lambda_0(\Xi_{**})\,\Xi_{**}^{\nu-1}\left[\frac{\Omega_{**}}{\tau(\Omega_{**})}\right]^2 d\Xi_{**}$$

The quantity $E_0=E_2-E_1$ does not depend on the upper limit of integration Ξ_*. Using (2.10) we finally obtain

$$E_0=\lim_{\Xi_*\to 0}(E_2-E_1)=\frac{cc_1^{2\nu}c_2}{\nu}>0 \tag{2.11}$$

The density distribution in the dust medium for $\tau=\tau'$ is represented by the parametric relation with parameter ξ_*.

$$\rho(\xi_*,\tau')=\frac{\gamma-1}{\gamma+1}\frac{A^{1/(\gamma-1)}\rho_0(\xi_*)}{\Omega_*^\nu}\frac{\tau^\nu(\Omega_*)}{(\tau')^\nu} \tag{2.12}$$

$$r(\xi_*,\tau')=\frac{\Omega_*\xi_*}{A^{1/\lambda}}\frac{\tau'}{\tau(\Omega_*)},\quad \xi_*<\xi_{\max}$$

For small r we have the asymptotic form

$$\rho(r,\ \tau') = k_1 r^\beta [1 + o(1)],\quad \beta = \frac{\gamma(\nu-2)-(3\nu-2)}{\gamma+1} \tag{2.13}$$

where k_1 is expressed by the constants introduced above. The value of $\rho(0, \tau')$ is finite only for $\nu = 3$ and $\gamma > 7$.

As $r \to r_{\max}$, the initial density $\rho(r, \tau')$ increases to infinity if $\gamma < (\nu+2)/\nu$, and decreases to zero if $\gamma > (\nu+2)/\nu$.

As we can see from (2.10), in the case of motions with plane waves $(\nu = 1)$, the density has a non-integrable singularity. After the explosion the gas pressure at the centre of symmetry vanishes for any value of ν. In one special case the solution of our problem is described by simple formulae. Let $\gamma = (\nu+2)/\nu$, $\tau' = 2$. Then

$$\Omega = [(\tau-1)^2 - 1]^{1/2},\quad \Xi_* = \left(\frac{\tau-2}{\tau}\right)^{1/(2\nu)} \tag{2.14}$$

$$r_* = \xi_{\max} A^{-1/2} \tau^a (\tau - 2)^b$$

$$p_0(\xi) = B\xi^\nu,\quad \rho_0(\xi) = \nu B \xi^{\nu-2},\quad B = \text{const}$$

$$\rho(r,\ \tau') = \frac{\nu B}{\nu+1}\left(\frac{A\xi_{\max}^{2\nu}}{4}\right)^{a/b} r^\omega\quad (r < r_{\max})$$

$$a = \frac{\nu-1}{2\nu},\quad b = \frac{\nu+1}{2\nu},\quad \omega = \frac{\nu-2-\nu^2}{\nu+1}$$

Let us now take $t' < 0$. In this case the sign of the constant A in (1.1) can be arbitrary.

If $A > 0$, we again introduce the variables Ω and τ given by formulae (2.5). On replacing $\xi_{\max}$ and $p_0(\xi_{\max})$ and the lower integration limit in (2.8) by ξ_s, $p_0(\xi_s)$ and Ω_s respectively, in (2.8) and (2.9) where $\Omega_s = \Omega(\tau'/2)$, $\xi_s = \xi_*(\Omega_s)$, $p_0(\xi_s) = \text{const} > 0$, we arrive at the expressions for the pressure and density distribution in the gas behind the shock wave. The law of motion of the discontinuity takes the form

$$\Xi_* = \left[\frac{\tau'\Omega_*}{2\Omega_s \tau(\Omega_*)}\right]^{(\gamma-1)/2} \tag{2.15}$$

The asymptotic formulae (2.10) and the expressions for the explosion energy (2.11) remain valid, taking the above replacements into account. Using (2.15) we can show that the shock wave goes to infinity in a finite time interval $(\tau', 0)$. The density distribution in the dust medium at the instant of explosion is determined as before by formulae (2.12), with $\xi_{\max} = \infty$.

Now let $A < 0$. We introduce the variables Ω and τ by formulae (2.5), replacing A by $-A$. Instead of (2.6) we shall have $d\Omega = \pm(\Omega^{-\lambda} - 1)^{1/2} d\tau$. In the time interval τ of length

$$\Delta = 2\int_0^1 (\Omega^{-\lambda} - 1)^{-1/2}\, d\Omega$$

the quantity Ω increases from 0 to 1 and again becomes 0. Here $\Omega^{\cdot}$ varies from $+\infty$ to $-\infty$. Consequently, the constraint $\tau' \geqslant -\Delta$ should be imposed on the instant of the explosion. The case of satisfying the strict inequality does not qualitatively differ from the case $A > 0$ above. If, however, $\tau' = -\Delta$, as $\tau \to 0$ the behavour of the solution changes. Thus, for the radius of the shock wave we can obtain the estimate

$$r_* \sim \tau^\alpha,\quad \alpha = \frac{4 - \nu(\gamma-1)^2}{2[2+\nu(\gamma-1)]}$$

It follows that as $\tau \to 0$, the shock wave goes to infinity if $\gamma > 1 + 2\nu^{-1/2}$, and returns to the centre of symmetry if $\gamma < 1 + 2\nu^{-1/2}$.

When $\gamma = (\nu+2)/\nu$, $\tau' = -\Delta = -2$ the solution has a simple form, and is obtained from (2.14) by replacing $A, \tau, \xi_{\max}$ by $(-A), (-\tau), \xi_s$ respectively, and arranging, where necessary, modular parentheses.

Notice that this solution and solution (2.14) (see /2/) could be obtained from the solution of the problem of an explosion in a medium of varying density $\rho = C_0 r^\omega$, where C_0 is a constant, and $\omega = (\nu - 2 - \nu^2)/(\nu+1)$, using a special group of the invariant transformations of the equations of gas dynamics, which corresponds to $\gamma = (\nu+2)/\nu$ (see /8, 9/).

3. Consider the example of Problem A from Section 1. Let $\gamma = (\nu+2)/\nu$, $\varepsilon = -1$, $G(\xi) \equiv C^2 = \text{const}$, $C > 0$. Such a choice corresponds to the Gaussian density and pressure distributions in the gas behind the shock wave,

$$\rho_0(\xi) = C_1 \exp\left(-\frac{\xi^2}{2C^2}\right),\quad p_0(\xi) = C^2 C_1 \exp\left(-\frac{\xi^2}{2C^2}\right)$$

We introduce new variables, $\Omega = (-A)^{1/2} R$, $\tau = -At$. On integrating (1.1) we obtain the relation

$$\Omega(\tau) = [(\tau - \tau_0)^2 + 1]^{1/2},\quad \tau_0 = \text{const}$$

Without loss of genrality we assume below that $\tau_0 = 0$.

From Eq. (1.6) we determine the law of motion for the shock wave,

$$\xi_* = C\nu^{-1/2}(\operatorname{arctg}\tau - \operatorname{arctg}\tau'), \quad r_* = (-A)^{-1/2}(\tau^2+1)^{1/2}\xi_* \tag{3.1}$$

where τ' is the instant of the shock wave emerging from the centre of symmetry.

It can be shown that $\lim_{\tau\to\tau'} E_2 = 0$ (E_2 is the total energy of the gas behind the shock wave). In the solution discussed, therefore, the formation of a discontinuity is not connected with the emission of energy at the instant τ'.

From (1.4) and (3.1) we find

$$v_1(\tau) = C\left(-\frac{A}{\nu(\tau^2+1)}\right)^{1/2}[\tau(\operatorname{arctg}\tau - \operatorname{arctg}\tau') - \nu] \tag{3.2}$$

Assuming in this equality that $\tau = \tau'$, we obtain that $v_1(\tau') < 0$, that is the shock wave emerges as a consequence of the collision between the dust particles at the instant τ'.

We shall require that after the formation of a discontinuity in the dust medium no caustics, i.e. surfaces of infinite density, appear. This requirement means that in the interval (τ', ∞) there should be a strictly increasing function $g(\tau) = r_*(\tau) + v_1(\tau)(\tau - \tau')/A$. It can be shown using relations (3.1) and (3.2) that for this it is necessary and sufficient to satisfy the inequality $\tau' > \tau_m'$, where τ_m' is the root of the equation $(\nu+1)\tau = \pi/2 - \operatorname{arctg}\tau$.

The solution for the dust is presented in the parametric form

$$\rho(\tau_*, \tau) = \frac{(-A)^{\nu/2}\rho_0(\xi_*)}{(\nu+1)\Omega_*^{\nu}} \frac{r_*^{\nu-1}(\tau_*)}{r^{\nu-1}(\tau_*, \tau)}\left[(-A)\frac{dr_*}{d\tau_*} - v_1(\tau_*)\right] \times$$
$$\left[(-A)\frac{dr_*}{d\tau_*} - v_1(\tau_*) - \frac{dv_1}{d\tau_*}(\tau_* - \tau)\right]^{-1}$$
$$r(\tau_*, \tau) = r_*(\tau_*) + v_1(\tau_*)(\tau_* - \tau)A^{-1}, \quad \tau < \tau_*$$
$$r_*(\tau_*) = (-A)^{-1/2}\Omega_*\xi_*, \quad \Omega_* = \Omega(\tau_*), \quad \xi_* = \xi_*(\tau_*)$$

The shock wave 'freezes' on the dust particles which at the instant τ' were at a distance

$$r_{\max} = C(-\nu A)^{-1/2}[\tau'(\pi/2 - \operatorname{arctg}\tau') + \nu]$$

from the centre of symmetry.

The solution obtained describes the collapse of cavities in the dust cloud during the formation of the shock wave.

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